

4.7 – Change of Basis

Note: Recall that $(\mathbf{v})_B$ is the coordinate vector for a vector \mathbf{v} relative to a basis B . When considered as a column vector, this author uses the notation $[\mathbf{v}]_B$.

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a finite-dimensional vector space V , and if $(\mathbf{v})_S = (c_1, c_2, \dots, c_n)$ is the coordinate vector of \mathbf{v} relative to S , then the mapping $\mathbf{v} \rightarrow (\mathbf{v})_S$ is the **coordinate map relative to S** from V to \mathbb{R}^n .

The standard basis for M_{22} is

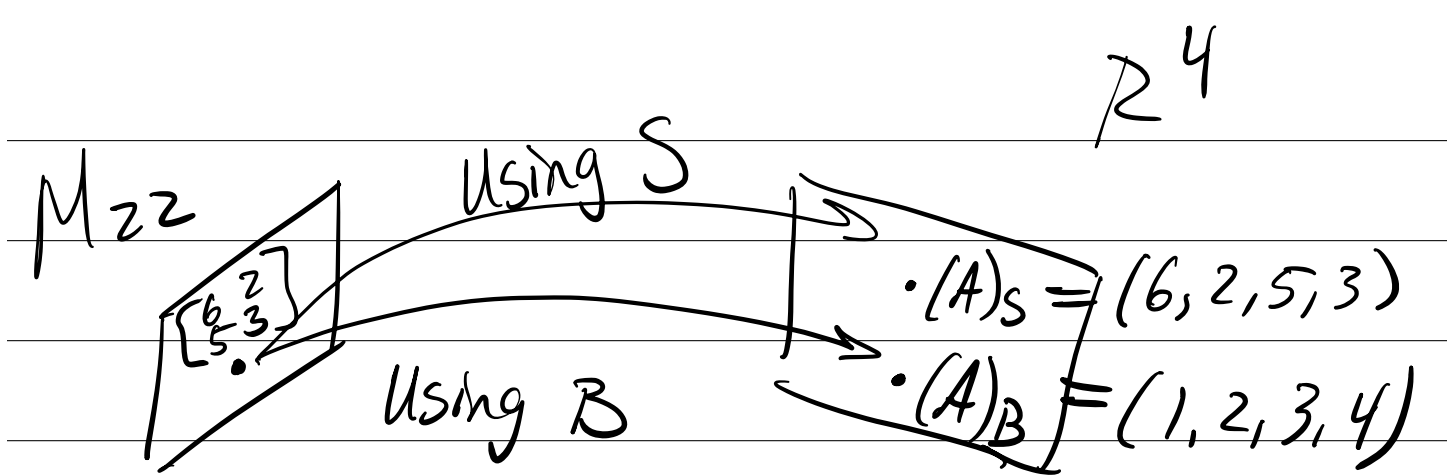
$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Relative to this basis, the coordinate vector for $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ is $(A)_S = (6, 2, 5, 3)$.

But as we saw in 4.5, the coordinate vector relative to $B = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is $(A)_B = (1, 2, 3, 4)$.

The coefficients change because the basis is different.

This produces a mapping from M_{22} into \mathbb{R}^4 :



The goal of this lesson is to provide a link between $(A)_S$ and $(A)_B$ or more generally, between $(\vec{v})_B$ and $(\vec{v})_{B'}$.

To transition from one basis B to another basis B' for the same vector space, we use a transition matrix, denoted by $P_{B \rightarrow B'}$.

In 4.5, we found $(\vec{v})_{B'}$ for $\vec{v} = (-1, 7, 2)$ with respect to the basis $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (2, -1, -1)$, $\mathbf{v}_2 = (-2, 1, 2)$, and $\mathbf{v}_3 = (3, 5, 4)$ by row reducing an augmented matrix. Here we achieve the same result by using a transition matrix from B , the standard basis for \mathbb{R}^3 , to B' .

To find a relationship between B and B' , we start by finding a linear combination of vectors in B' that maps to vectors in B .

$$a \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + b \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

this is 3 systems that we can solve simultaneously:

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 3 & 1 & 0 & 0 \\ -1 & 1 & 5 & 0 & 1 & 0 \\ -1 & 2 & 4 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6/13 & -14/13 & 1 \\ 0 & 1 & 0 & 1/13 & -11/13 & 1 \\ 0 & 0 & 1 & 1/13 & 2/13 & 0 \end{array} \right]$$

Columns are vectors in B'

Columns are vectors in B

maps B' to \vec{e}_1 ... \vec{e}_2 ... \vec{e}_3

The resulting matrix on the right maps a vector relative to B to a vector relative to B' .

Consider $\vec{v} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$. This is $[\vec{v}]_B$

$$\text{Now: } \begin{bmatrix} 6/13 & -14/13 & 1 \\ 1/13 & -11/13 & 1 \\ 1/13 & 2/13 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ 1 \end{bmatrix} = [\vec{v}]_{B'}$$

$$\text{thus, } \begin{bmatrix} -6 \\ -4 \\ 1 \end{bmatrix} = [\vec{v}]_{B'} = \begin{bmatrix} 6/13 & -14/13 & 1 \\ 1/13 & -11/13 & 1 \\ 1/13 & 2/13 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$

$$[\vec{v}]_{B'} = P_{B \rightarrow B'} [\vec{v}]_B$$

Definition: If we change the basis for a vector space V from an old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n\}$, then $[\mathbf{u}_i]_{B'}$ are the coordinate vectors for the old basis vectors relative to the new basis. The **transition matrix from B to B'** is the matrix having these vectors as its columns and is written [by this author] as $P_{B \rightarrow B'} = [[\mathbf{u}_1]_{B'} | [\mathbf{u}_2]_{B'} | \dots | [\mathbf{u}_n]_{B'}]$. Note that this is a partitioned matrix.

We use this as follows: For every vector \mathbf{v} in V ,

$$[\mathbf{v}]_{B'} = P_{B \rightarrow B'} [\mathbf{v}]_B$$

#3 Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\}$ for R^3 , where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{u}'_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \mathbf{u}'_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \mathbf{u}'_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

a. Find the transition matrix B to B' .

find $P_{B \rightarrow B'}$ that is,

b. Compute the coordinate vector $([\mathbf{w}]_B)$ where $\mathbf{w} = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$ and use the transition

matrix in part (a) to compute $[\mathbf{w}]_{B'}$.

c. Check your work by computing $[\mathbf{w}]_{B'}$ directly.

$$a) \left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & 5/2 \\ 0 & 1 & 0 & -2 & -3 & -1/2 \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array} \right]$$

new basis vectors (to)

old basis vectors (from)

$P_{B \rightarrow B'}$

$$P_{B \rightarrow B'} = \begin{bmatrix} 3 & 2 & 5/2 \\ -2 & -3 & -1/2 \\ 5 & 1 & 6 \end{bmatrix}$$

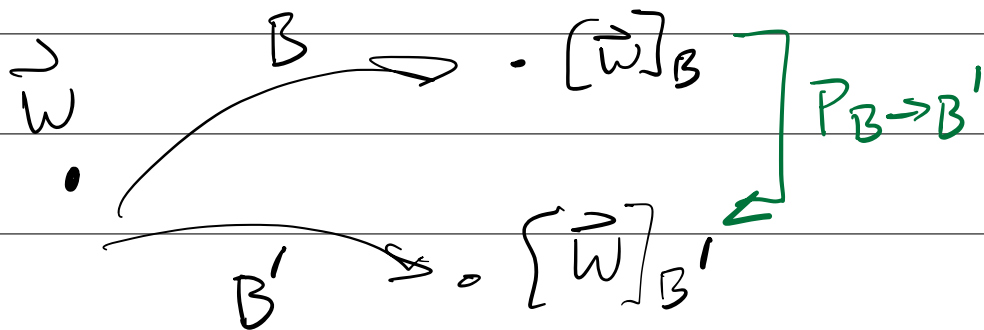
vectors in B

$$b) \left[\begin{array}{ccc|c} 2 & 2 & 1 & -5 \\ 1 & -1 & 2 & 8 \\ 1 & 1 & 1 & -5 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -5 \end{array} \right]$$

$$(9\vec{u}_1 - 9\vec{u}_2 - 5\vec{u}_3 = \vec{w}) \quad [\vec{w}]_B = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$$

$$[\vec{w}]_{B'} = \begin{bmatrix} 3 & 2 & 5/2 \\ -2 & -3 & -1/2 \\ 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$$

$$[\vec{w}]_{B'} = \begin{bmatrix} -7/2 \\ 23/2 \\ 6 \end{bmatrix}$$



check:

$$c) \left[\begin{array}{ccc|c} 3 & 1 & -1 & -5 \\ 1 & 1 & 0 & 8 \\ -5 & -3 & 2 & -5 \end{array} \right] \text{ yields } [\vec{w}]_{B'} = \begin{bmatrix} -7/2 \\ 23/2 \\ 6 \end{bmatrix}$$

Theorem 4.7.1 If P is the transition matrix from a basis B to a basis B' for a finite-dimensional vector space V , then P is invertible, and P^{-1} is the transition matrix from B' to B .

#13 If P is the transition matrix from a basis B' to a basis B , and Q is the transition matrix from B to a basis C , what is the transition matrix from B' to C ? What is the transition matrix from C to B' ?

